DISPERSION RELATIONS

A MINI-REVIEW
B. GRINSTEIN
Outline

* Formalism
* Recipe
* Errors
* Uses
Define

\[ \Pi_{J}^{\mu\nu}(q) = \frac{1}{q^2}(q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{J}^{T}(q^2) + \frac{q^\mu q^\nu}{q^2} \Pi_{J}^{L}(q^2) \equiv i \int d^4x e^{iqx} \langle 0| T J^\mu(x) J^{\nu}(0)|0 \rangle. \]

Dispersion relations

\[ \chi_{J}^{L}(q^2) \equiv \frac{1}{\pi} \int_{0}^{\infty} dt \frac{\text{Im} \, \Pi_{J}^{L}(t)}{(t - q^2)^2} \]

\[ \chi_{J}^{T}(q^2) \equiv \frac{1}{2} \frac{\text{Im} \, \Pi_{J}^{T}(t)}{(t - q^2)^3} \]

Inequality

\[ \text{Im} \, \Pi_{j}^{T:L} = \frac{1}{2} \sum_{X} (2\pi)^4 \delta^4(q - p_{X}) |\langle 0|J|X \rangle|^2 \geq \pi (2\pi)^3 \delta^4(q - p_{B} - p_{\pi}) |\langle 0|J|B\pi \rangle|^2 \]

Related by crossing to decay form factor

\[ \int_{t+}^{\infty} dt \frac{W(t) \, |F(t)|^2}{(t - q^2)^3} \leq 1 \]

\[ t_+ = (m_B + m_\pi)^2 \]
Complex magic

$$z(t) = \frac{t_- - t}{(\sqrt{t_+ - t} + \sqrt{t_+ - t_-})^2}$$

$$t_\pm \equiv (t_+ - t_-)^2$$

Form factor for $B \to \pi \ell \nu$

$$F(t) = \frac{1}{P(t) \phi(t)} \sum_{n=0}^{\infty} a_n z^n$$

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1$$
\[ F(t) = \frac{1}{P(t)\phi(t)} \sum_{n=0}^{\infty} a_n z^n \]

\[ \sum_{n=0}^{\infty} |a_n|^2 \leq 1 \]

Blaschke Factor: removes poles, needs location of poles \((B\bar{c} \text{ masses or } B^* \text{ mass})\)

Outer function: phase space, \(\chi(0)\) (QCD), jacobian

Range of \(z\) very small, eg, zero to 0.07 for \(b \to c\)
Recipe


\[ \phi_i(t; t_0) = \sqrt{\frac{n_I}{K \pi \chi}} \left( \frac{t_+ - t}{t_+ - t_0} \right)^{\frac{1}{4}} \left( \sqrt{t_+ - t + \sqrt{t_+ - t_0}} (t_+ - t)^{\frac{a}{2}} \right. \]

\[ \times \left( \sqrt{t_+ - t + \sqrt{t_+ - t_0}} \right)^{\frac{b}{2}} \left( \sqrt{t_+ - t + \sqrt{t_+}} \right)^{-(c+3)}. \]

For example, outer function ...

<table>
<thead>
<tr>
<th>( F_i )</th>
<th>( K )</th>
<th>( \chi )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_+ )</td>
<td>48</td>
<td>( \chi^T(+u) )</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( f_0 )</td>
<td>16</td>
<td>( \chi^L(+u) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( f )</td>
<td>24</td>
<td>( \chi^T(-u) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \mathcal{F}_1 )</td>
<td>48</td>
<td>( \chi^T(-u) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( g )</td>
<td>96</td>
<td>( \chi^T(+u) )</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( \mathcal{F}_2 )</td>
<td>64</td>
<td>( \chi^L(-u) )</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1. Factors entering Eq. (4.14) or (4.23) for the meson form factors \( F_i \) in \( B \to D^{(*)} \).

<table>
<thead>
<tr>
<th>( F_i )</th>
<th>( K )</th>
<th>( \chi )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_0 )</td>
<td>8</td>
<td>( \chi^L(+u) )</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( F_1 )</td>
<td>12</td>
<td>( \chi^T(+u) )</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( H_V )</td>
<td>24</td>
<td>( \chi^T(+u) )</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( G_0 )</td>
<td>8</td>
<td>( \chi^L(-u) )</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( G_1 )</td>
<td>12</td>
<td>( \chi^T(-u) )</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( H_A )</td>
<td>24</td>
<td>( \chi^T(-u) )</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2. Factors entering Eq. (4.14) or (4.23) for the baryon form factors \( F_i \).
Errors

Just listed. Analysis published. Errors small. What do you expect in 10 min?

Perturbative OPE and mb: $\chi_L(0)$

Masses for Blaschke factor: $P(z)$

Truncation error:

$$F(t) = \frac{1}{P(t)\phi(t)} \sum_{n=0}^{\infty} a_n z^n$$

Not an error if you can (get upper lower bounds)

Additional assumptions (strengthen bound) introduce additional errors. Typically assume symmetry, combine form factors: isospin, SU(3), HQS, etc.
Uses

Heavy-to heavy: Old hat. Large discrepancy between QCD based analysis and linear extrapolations for $F(1)$ and slope ($F'(1)$)

\[ V_{c\bar{s}}(F(w)) \]

\[ \rho^2 \]

\[ |V_{c\bar{s}}| F_s(1) \times 10^3 \]

\[ |V_{c\bar{s}}| F(1) \times 10^2 \]
Heavy-to-light: New use. $B \rightarrow \pi \ell \nu$ Arnesen, BG, Rothstein, Setwart (to appear)

Upper and lower bounds on form factors from fitting to:
1. Lattice points for $q^2 > 16$ GeV$^2$ (absolute normalization)
2. Lever arm from SCET: factorization in $B \rightarrow \pi \pi$ at $q^2 = 0$ (gives $|V_{ub}| f_+(0)$, so bootstrap, i.e., fit)

Graph showing $f_+(q^2)$ with curves for SCET and Lattice (FNAL, HPQCD)
Relevance of $q^2 = 0$ point

\[(q^2 - m_{B^*}^2) f_+(q^2)\]

Comparing (FNAL):

\[|V_{ub}|_{(q^2 \geq 16 \text{GeV}^2)} = (3.87 \pm 0.70 \pm 0.22^{+0.62}_{-0.48}) \times 10^{-3}\]

Fixing from "better behaved" form factor?

Four lattice points

Exp.

Theory: dominated by input points uncertainty

$|V_{ub}| = (4.08 \pm 0.22 \pm 0.40) \times 10^{-3}$
Provocative

* Use technique above (AGRS) to show inadequacy of BK ff
* Improved parameterization (in progress)
* Others? e.g., $B \rightarrow \rho \ell \nu, B \rightarrow A_1 \ell \nu$?
* Find parametrizations from CLEO-c?
* Partially-inclusives?? (shape function)
* Include truncation error (ie, range) in systematics of heavy-to-heavy $F(1)$ and $F'(1)$
Use only three lattice points